

Tight triangulations of some 4-manifolds

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Abstract.

Walkup's class $\mathcal{K}(d)$ consists of the d -dimensional simplicial complexes all whose vertex links are stacked $(d-1)$ -spheres. According to a result of Walkup, the face vector of any triangulated 4-manifold X with Euler characteristic χ satisfies $f_1 \geq 5f_0 - \frac{15}{2}\chi$, with equality only for $X \in \mathcal{K}(4)$. Kühnel observed that this implies $f_0(f_0 - 11) \geq -15\chi$, with equality only for 2-neighborly members of $\mathcal{K}(4)$. For $n = 6, 11$ and 15 , there are triangulated 4-manifolds with $f_0 = n$ and $f_0(f_0 - 11) = -15\chi$. In this article, we present triangulated 4-manifolds with $f_0 = 21, 26$ and 41 which satisfy $f_0(f_0 - 11) = -15\chi$. All these triangulated manifolds are tight and strongly minimal.

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1 Introduction

Walkup's class $\mathcal{K}(d)$ consists of the d -dimensional simplicial complexes all whose vertex links are stacked $(d-1)$ -spheres. Kalai showed that for $d \geq 4$, all connected members of $\mathcal{K}(d)$ are obtained from stacked d -spheres by finitely many elementary handle additions (cf. Proposition 2.4 below). According to a result of Walkup [10], the face vector $(f_0, f_1, f_2, f_3, f_4)$ of any triangulated 4-manifold X with Euler characteristic χ satisfies $f_1 \geq 5f_0 - \frac{15}{2}\chi$, with equality only for $X \in \mathcal{K}(4)$. Kühnel [7] observed that this implies $f_0(f_0 - 11) \geq -15\chi$, with equality only for 2-neighborly members of $\mathcal{K}(4)$. Clearly, for the equality, $f_0 \equiv 0, 5, 6, 11 \pmod{15}$. For $n = 6, 11$ and 15 , there are such triangulated manifolds with $f_0 = n$, namely, the 6-vertex standard 4-sphere S_6^4 , the unique 11-vertex triangulation of $S^3 \times S^1$ of Kühnel and the 15-vertex triangulation of $(S^3 \times S^1)^{\#3}$ obtained by Bagchi and Datta [1]. Recently, the second author [9] found ten 15-vertex triangulations of $(S^3 \times S^1)^{\#3}$ and one more 15-vertex triangulation of $(S^3 \times S^1)^{\#3}$.

Observe that if $f_0(f_0 - 11) = -15\chi$ and $f_0 \geq 15$ then χ is even and negative. Moreover, $-\chi/2$ divides f_0 if and only if $f_0 = 21, 26$ or 41 . Note that, in each of the three cases, $p = -\chi/2$ is a prime. For these cases, we have constructed triangulated 4-manifolds which satisfy $f_0(f_0 - 11) = -15\chi$ and have automorphism groups \mathbb{Z}_p . More explicitly, we have constructed a 21-vertex triangulation of $(S^3 \times S^1)^{\#8}$, a 21-vertex triangulation of $(S^3 \times S^1)^{\#8}$, a 26-vertex triangulation of $(S^3 \times S^1)^{\#14}$ and a 41-vertex triangulation of $(S^3 \times S^1)^{\#42}$. For each of our triangulated manifolds, the full automorphism group is \mathbb{Z}_p , where $p = -\chi/2$.

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Effenberger proved that any 2-neighborly \mathbb{F} -orientable member of $\mathcal{K}(4)$ is \mathbb{F} -tight (cf. Proposition 2.5 below). By a result (Proposition 2.6 below) of Bagchi and Datta, for any field \mathbb{F} , any \mathbb{F} -tight member of $\mathcal{K}(4)$ is strongly minimal. Therefore, our orientable (resp., non-orientable) examples are \mathbb{Q} -tight (resp., \mathbb{Z}_2 -tight) and strongly minimal.

2 Preliminaries

All simplicial complexes considered here are finite and abstract. By a triangulated manifold/sphere/ball, we mean an abstract simplicial complex whose geometric carrier is a topological manifold/sphere/ball. We identify two complexes if they are isomorphic. A d -dimensional simplicial complex is called *pure* if all its maximal faces (called *facets*) are d -dimensional. A d -dimensional pure simplicial complex is said to be a *weak pseudomanifold* if each of its $(d-1)$ -faces is in at most two facets. For a d -dimensional weak pseudomanifold X , the *boundary* ∂X of X is the pure subcomplex of X whose facets are those $(d-1)$ -dimensional faces of X which are contained in unique facets of X . The *dual graph* $\Lambda(X)$ of a pure simplicial complex X is the graph whose vertices are the facets of X , where two facets are adjacent in $\Lambda(X)$ if they intersect in a face of codimension one. A *pseudomanifold* is a weak pseudomanifold with a connected dual graph. All connected triangulated manifolds are automatically pseudomanifolds.

If X is a d -dimensional simplicial complex then, for $0 \leq j \leq d$, the number of its j -faces is denoted by $f_j = f_j(X)$. The vector $f(X) := (f_0, \dots, f_d)$ is called the *face vector* of X and the number $\chi(X) := \sum_{i=0}^d (-1)^i f_i$ is called the *Euler characteristic* of X . As is well known, $\chi(X)$ is a topological invariant, i.e., it depends only on the homeomorphic type of $|X|$. A simplicial complex X is said to be *l -neighborly* if any l vertices of X form a face of X . A 2-neighborly simplicial complex is also called a *neighborly* simplicial complex.

A *standard d -ball* is a pure d -dimensional simplicial complex with one facet. The standard ball with facet σ is denoted by $\bar{\sigma}$. A d -dimensional pure simplicial complex X is called a *stacked d -ball* if there exists a sequence B_1, \dots, B_m of pure simplicial complexes such that B_1 is a standard d -ball, $B_m = X$ and, for $2 \leq i \leq m$, $B_i = B_{i-1} \cup \bar{\sigma}_i$ and $B_{i-1} \cap \bar{\sigma}_i = \bar{\tau}_i$, where σ_i is a d -face and τ_i is a $(d-1)$ -face of σ_i . Clearly, a stacked ball is a pseudomanifold. A simplicial complex is called a *stacked d -sphere* if it is the boundary of a stacked $(d+1)$ -ball. A trivial induction on m shows that a stacked d -ball actually triangulates a topological d -ball, and hence a stacked d -sphere is a triangulated d -sphere. If X is a stacked ball then clearly $\Lambda(X)$ is a tree. So, a stacked ball is a pseudomanifold whose dual graph is a tree. But, the converse is not true (e.g., the 3-pseudomanifold X whose facets are 1234, 2345, 3456, 4567, 5671 is a pseudomanifold for which $\Lambda(X)$ is a tree but $|X|$ is not a ball). Here we have

Lemma 2.1. *Let X be a pure d -dimensional simplicial complex.*

- (i) *If $\Lambda(X)$ is a tree then $f_0(X) \leq f_d(X) + d$.*
- (ii) *$\Lambda(X)$ is a tree and $f_0(X) = f_d(X) + d$ if and only if X is a stacked ball.*

Proof. Let $f_d(X) = m$ and $f_0(X) = n$. So, $\Lambda(X)$ is a graph with m vertices. We prove (i) by induction on m . If $m = 1$ then the result is true with equality. So, assume that $m > 1$ and the result is true for smaller values of m . Since $\Lambda(X)$ is a tree, it has a vertex σ of degree one (leaf) and hence $\Lambda(X) - \sigma$ is again a tree. Let Y be the pure simplicial complex (of dimension d) whose facets are those of X other than σ . Since σ has a $(d-1)$ -face in

Y , it follows that $f_0(Y) \geq n - 1$. Since $f_d(Y) = m - 1$, the result is true for Y and hence $f_0(Y) \leq (m - 1) + d$. Therefore, $n \leq f_0(Y) + 1 \leq 1 + (m - 1) + d = m + d$. This proves (i).

If X is a stacked d -ball with m facets then X is a pseudomanifold and by the definition (since at each of the $m - 1$ stages one adds one facet and one vertex), $n = (d + 1) + (m - 1) = m + d$. Conversely, let $\Lambda(X)$ be a tree and $n = f_0(X) = m + d$. Let Y be as above. Since $f_0(Y) \geq n - 1$, it follows that $f_0(Y) = n$ or $n - 1$. If $f_0(Y) = n$ then $f_0(Y) = n > (m - 1) + d = f_d(Y) + m$, a contradiction to part (i). So, $f_0(Y) = n - 1$ and hence $Y \cap \bar{\sigma}$ is a $(d - 1)$ -face of σ . Since $f_d(Y) = m - 1$, by induction hypothesis, Y is a stacked d -ball and hence $X = Y \cup \bar{\sigma}$ is a stacked d -ball. This proves (ii). \square

Corollary 2.2. *Let X be a pure d -dimensional simplicial complex and let CX denote a cone over X . Then CX is a stacked $(d + 1)$ -ball if and only if X is a stacked d -ball.*

Proof. Notice that $f_{d+1}(CX) = f_d(X)$ and $f_0(CX) = f_0(X) + 1$. Also $\Lambda(CX)$ is naturally isomorphic to $\Lambda(X)$. The proof now follows from Lemma 2.1. \square

In [10], Walkup defined the class $\mathcal{K}(d)$ as the family of all d -dimensional simplicial complexes all whose vertex-links are stacked $(d - 1)$ -spheres. Clearly, all the members of $\mathcal{K}(d)$ are triangulated closed manifolds. Let $\mathcal{K}^*(d)$ be the class of 2-neighborly members of $\mathcal{K}(d)$. We know the following.

Proposition 2.3 (Bagchi and Datta [2]). *Let M be a connected closed triangulated manifold of dimension $d \geq 3$. Let $\beta_1 = \beta_1(M; \mathbb{Z}_2)$. Then the face vector of M satisfies:*

$$(a) \quad f_j \geq \begin{cases} \binom{d+1}{j} f_0 + j \binom{d+2}{j+1} (\beta_1 - 1), & \text{if } 1 \leq j < d, \\ df_0 + (d - 1)(d + 2)(\beta_1 - 1), & \text{if } j = d. \end{cases}$$

$$(b) \quad \binom{f_0 - d - 1}{2} \geq \binom{d+2}{2} \beta_1.$$

When $d \geq 4$, the equality holds in (a) (for some $j \geq 1$, equivalently, for all j) if and only if $M \in \mathcal{K}(d)$, and equality holds in (b) if and only if $M \in \mathcal{K}^*(d)$.

The case $d = 4$ of the above proposition is due to Walkup [10] and Kühnel [7]. Part (b) of the above proposition is due to Lutz, Sulanke and Swartz [8].

Proposition 2.4 (Kalai [6]). *For $d \geq 4$, a connected simplicial complex X is in $\mathcal{K}(d)$ if and only if X is obtained from a stacked d -sphere by $\beta_1(X)$ combinatorial handle additions. In consequence, any such X triangulates either $(S^{d-1} \times S^1)^{\# \beta_1}$ or $(S^{d-1} \times S^1)^{\# \beta_1}$ according as X is orientable or not. (Here $\beta_1 = \beta_1(X)$.)*

It follows from Proposition 2.4 that

$$\chi(X) = 2 - 2\beta_1(X) \quad \text{for } X \in \mathcal{K}(d). \quad (1)$$

For a field \mathbb{F} , a d -dimensional simplicial complex X is called *tight with respect to \mathbb{F}* (or \mathbb{F} -tight) if (i) X is connected, and (ii) for all induced subcomplexes Y of X and for all $0 \leq j \leq d$, the morphism $H_j(Y; \mathbb{F}) \rightarrow H_j(X; \mathbb{F})$ induced by the inclusion map $Y \hookrightarrow X$ is injective. If X is \mathbb{Q} -tight then it is \mathbb{F} -tight for all fields \mathbb{F} and called *tight* (cf. [3]).

A d -dimensional simplicial complex X is called *minimal* if $f_0(X) \leq f_0(Y)$ for every triangulation Y of the geometric carrier $|X|$ of X . We say that X is *strongly minimal* if $f_i(X) \leq f_i(Y)$, $0 \leq i \leq d$, for all such Y . We know the following.

Proposition 2.5 (Effenberger [4], Bagchi and Datta [2]). *Every \mathbb{F} -orientable member of $\mathcal{K}^*(d)$ is \mathbb{F} -tight for $d \neq 3$. An \mathbb{F} -orientable member of $\mathcal{K}^*(3)$ is \mathbb{F} -tight if and only if $\beta_1(X) = (f_0(X) - 4)(f_0(X) - 5)/20$.*

Proposition 2.6 (Bagchi and Datta [2]). *Every \mathbb{F} -tight member of $\mathcal{K}(d)$ is strongly minimal.*

Let $\overline{\mathcal{K}}(d)$ be the class of all d -dimensional simplicial complexes all whose vertex-links are stacked $(d-1)$ -balls. Clearly, if $N \in \overline{\mathcal{K}}(d)$ then N is a triangulated manifold with boundary and satisfies

$$\text{skel}_{d-2}(N) = \text{skel}_{d-2}(\partial N). \quad (2)$$

Here $\text{skel}_j(N) = \{\alpha \in N : \dim(\alpha) \leq j\}$ is the j -skeleton of N . We know the following.

Proposition 2.7 (Bagchi and Datta [3]). *For $d \geq 4$, $M \mapsto \partial M$ is a bijection from $\overline{\mathcal{K}}(d+1)$ to $\mathcal{K}(d)$.*

Corollary 2.8. *For $d \geq 4$, if $M \in \overline{\mathcal{K}}(d+1)$ then $\text{Aut}(M) = \text{Aut}(\partial M)$.*

Proof. Clearly $\text{Aut}(M) \subseteq \text{Aut}(\partial M)$. If $\sigma : V(M) \rightarrow V(M)$ is in $\text{Aut}(\partial M)$ then $\sigma(M) \in \overline{\mathcal{K}}(d+1)$ and $\partial(\sigma(M)) = \sigma(\partial M) = \partial M$. Therefore by Proposition 2.7, $\sigma(M) = M$. This implies $\sigma \in \text{Aut}(M)$. Therefore, $\text{Aut}(\partial M) \subseteq \text{Aut}(M)$ and hence $\text{Aut}(M) = \text{Aut}(\partial M)$. \square

3 Examples

Example 3.1. Let $V_{21} = \cup_{i=0}^6 \{a_i, b_i, c_i\}$ be a set of 21 elements. Let the cyclic group \mathbb{Z}_7 act on V_{21} as $i \cdot a_j = a_{i+j}$, $i \cdot b_j = b_{i+j}$ and $i \cdot c_j = c_{i+j}$ (additions being modulo 7). Consider the pure 5-dimensional simplicial complex A_{21}^5 on the vertex-set V_{21} as follows. Modulo the group \mathbb{Z}_7 the facets are

$$\begin{aligned} \sigma_0 &= a_0 a_1 a_2 b_0 b_1 c_0, \kappa_0 = a_1 a_2 b_0 b_1 b_2 c_0, \tau_0 = a_1 a_2 a_3 b_0 b_1 b_2, \alpha_0 = a_0 a_1 b_0 b_1 c_0 c_3, \\ \beta_0 &= a_0 a_1 b_0 b_3 c_0 c_3, \mu_0 = a_0 b_0 b_3 c_0 c_3 c_4, \nu_0 = a_0 a_3 b_3 c_0 c_3 c_4, \gamma_0 = a_3 b_3 c_0 c_3 c_4 c_6. \end{aligned}$$

The full list of 56 facets can be obtained by applying the group \mathbb{Z}_7 to these eight facets. The dual graph of A_{21}^5 is the union of two 21-cycles $C_1 = \sigma_0 \kappa_0 \tau_0 \sigma_1 \kappa_1 \tau_1 \cdots \sigma_6 \kappa_6 \tau_6 \sigma_0$, $C_2 = \mu_0 \nu_0 \gamma_0 \mu_3 \nu_3 \gamma_3 \cdots \mu_4 \nu_4 \gamma_4 \mu_0$ and paths $P_i = \sigma_i \alpha_i \beta_i \mu_i$ for $i \in \mathbb{Z}_7$. It can be shown that A_{21}^5 is a neighborly member of $\overline{\mathcal{K}}(5)$ (see Lemma 4.2 below). Let $M_{21}^4 := \partial A_{21}^5$. Then $M_{21}^4 \in \mathcal{K}^*(4)$ and hence, by Proposition 2.3, $\chi(M_{21}^4) = -14$. Then by (1), $\beta_1(M_{21}^4) = 8$. One can show that M_{21}^4 is orientable (by giving an explicit orientation or using `simpcomp` [5]) and so, by Proposition 2.4, M_{21}^4 triangulates $(S^3 \times S^1)^{\#8}$.

Example 3.2. Let V_{21} be the vertex-set with group \mathbb{Z}_7 acting on it as in Example 3.1. Consider the pure 5-dimensional simplicial complex B_{21}^5 whose facets modulo \mathbb{Z}_7 action described above are

$$\begin{aligned} \sigma_0 &= a_0 a_1 a_2 b_0 b_1 c_0, \kappa_0 = a_0 a_1 a_2 b_1 b_2 c_0, \tau_0 = a_0 a_1 a_2 a_3 b_1 b_2, \alpha_0 = a_0 a_1 b_0 b_1 c_0 c_3, \\ \beta_0 &= a_0 b_0 b_1 b_3 c_0 c_3, \mu_0 = a_0 b_0 b_3 c_0 c_3 c_4, \nu_0 = a_3 b_0 b_3 c_0 c_3 c_4, \gamma_0 = a_3 b_3 c_0 c_3 c_4 c_6. \end{aligned}$$

The dual graph of B_{21}^5 is the same as that of A_{21}^5 . It can be shown that B_{21}^5 is a neighborly member of $\overline{\mathcal{K}}(5)$ (see Lemma 4.2 below). Let $N_{21}^4 := \partial B_{21}^5$. Then $N_{21}^4 \in \mathcal{K}^*(4)$ and hence, by Proposition 2.3, $\chi(N_{21}^4) = -14$. Then by (1), $\beta_1(N_{21}^4) = 8$. Using `simpcomp`, one can check that N_{21}^4 is non-orientable and so, by Proposition 2.4, it triangulates $(S^3 \times S^1)^{\#8}$.

Example 3.3. Let $V_{26} = \cup_{i=0}^{12} \{a_i, b_i\}$ be a set of 26 elements. The cyclic group \mathbb{Z}_{13} acts on V_{26} as $i \cdot a_j = a_{i+j}$, $i \cdot b_j = b_{i+j}$ (additions being modulo 13). Consider the 5-dimensional pure simplicial complex B_{26}^5 on the vertex-set V_{26} whose facets modulo the group \mathbb{Z}_{13} are

$$\begin{aligned}\sigma_0 &= a_0 a_{10} a_{11} a_{12} b_9 b_{10}, \tau_0 = a_0 a_1 a_{10} a_{11} a_{12} b_{10}, \alpha_0 = a_0 a_{11} a_{12} b_5 b_9 b_{10}, \\ \beta_0 &= a_0 a_{11} a_{12} b_2 b_5 b_{10}, \gamma_0 = a_0 a_7 a_{12} b_2 b_5 b_{10}, \mu_0 = a_7 a_{12} b_0 b_2 b_5 b_{10}, \delta_0 = a_7 b_0 b_2 b_5 b_8 b_{10}.\end{aligned}$$

The full list of 91 facets can be obtained by applying the group \mathbb{Z}_{13} to these seven facets. The dual graph of B_{26}^5 is the union of two 26-cycles $C_1 = \sigma_0 \tau_0 \sigma_1 \tau_1 \cdots \sigma_{12} \tau_{12} \sigma_0$, $C_2 = \mu_0 \delta_0 \mu_8 \delta_8 \cdots \mu_5 \delta_5 \mu_0$ and paths $P_i = \sigma_i \alpha_i \beta_i \gamma_i \mu_i$ for $i \in \mathbb{Z}_{13}$. It can be shown that B_{26}^5 is a neighborly member of $\overline{\mathcal{K}}(5)$ (see Lemma 4.2 below). Let $N_{26}^4 := \partial B_{26}^5$. Then $N_{26}^4 \in \mathcal{K}^*(4)$ and hence, by Proposition 2.3, $\chi(N_{26}^4) = -26$. Then by (1), $\beta_1(N_{26}^4) = 14$. Using `simpcomp`, one can check that N_{26}^4 is non-orientable and so, by Proposition 2.4, N_{26}^4 triangulates $(S^3 \times S^1)^{\#14}$.

Example 3.4. Let $V_{41} = \{a_0, a_1, \dots, a_{40}\}$ be a set of 41 elements. The cyclic group \mathbb{Z}_{41} acts on V_{41} as $i \cdot a_j = a_{i+j}$ (addition is modulo 41). Consider the pure 5-dimension simplicial complex A_{41}^5 on the vertex-set V_{41} as follows. Modulo the group \mathbb{Z}_{41} its facets are

$$\begin{aligned}\sigma_0 &= a_{36} a_{37} a_{38} a_{39} a_{40} a_0, \alpha_0 = a_{36} a_{37} a_{38} a_{39} a_0 a_6, \beta_0 = a_{37} a_{38} a_{39} a_0 a_6 a_{13}, \\ \gamma_0 &= a_{38} a_{39} a_0 a_6 a_{13} a_{20}, \delta_0 = a_{39} a_0 a_6 a_{13} a_{20} a_{27}, \mu_0 = a_6 a_{13} a_{20} a_{27} a_{34} a_0.\end{aligned}$$

The full list of 246 facets of A_{41}^5 may be obtained from these basic six facets applying the group \mathbb{Z}_{41} . The dual graph of A_{41}^5 is the union of two 41-cycles $C_1 = \sigma_0 \sigma_1 \cdots \sigma_{40} \sigma_0$, $C_2 = \mu_0 \mu_7 \mu_{14} \cdots \mu_{34} \mu_0$ and paths $P_i = \sigma_i \alpha_i \beta_i \gamma_i \delta_i \mu_i$ for $i \in \mathbb{Z}_{41}$. Then A_{41}^5 is a neighborly member of $\overline{\mathcal{K}}(5)$ (see Lemma 4.2 below). Let $M_{41}^4 := \partial A_{41}^5$. Then $M_{41}^4 \in \mathcal{K}^*(4)$ and hence, by Proposition 2.3, $\chi(M_{41}^4) = -82$. Therefore, by (1), $\beta_1(M_{41}^4) = 1 - \chi(M_{41}^4)/2 = 42$. One can check (by giving an explicit orientation or using `simpcomp`) that M_{41}^4 is orientable and hence, by Proposition 2.4, M_{41}^4 triangulates $(S^3 \times S^1)^{\#42}$.

For easy reference, we summarize the results of this section in table below. Notice that M_{41}^4 admits a vertex-transitive automorphism group.

M	$f_0(M)$	$\chi(M)$	$\beta_1(M)$	$\text{Aut}(M)$	$f(M)$	$ M $
M_{21}^4	21	-14	8	\mathbb{Z}_7	(21, 210, 490, 525, 210)	$(S^3 \times S^1)^{\#8}$
N_{21}^4	21	-14	8	\mathbb{Z}_7	(21, 210, 490, 525, 210)	$(S^3 \times S^1)^{\#8}$
N_{26}^4	26	-26	14	\mathbb{Z}_{13}	(26, 325, 780, 845, 338)	$(S^3 \times S^1)^{\#14}$
M_{41}^4	41	-82	42	\mathbb{Z}_{41}	(41, 820, 2050, 2255, 902)	$(S^3 \times S^1)^{\#42}$

Table 1: Summary of results of Section 3

4 Construction Details

Let X be a neighborly member of $\overline{\mathcal{K}}(d)$. Then all vertex-links, and equivalently vertex-stars in X are stacked balls. By Corollary 2.2, we see that the facets containing a given vertex x form an $(f_0(X) - d)$ -vertex induced subtree of $\Lambda(X)$. Thus for each vertex, we get a subtree of $\Lambda(X)$ (namely, the dual graph of $\text{st}_X(x)$). From the neighborliness of X , it follows that any two of these trees intersect. Now we invert the question, i.e, given a graph G and an

intersecting family \mathcal{T} of induced subtrees of G , can we get a neighborly member of $\overline{\mathcal{K}}(d)$? Our next lemma answers this in affirmative under certain conditions. Given a graph G and a family $\mathcal{T} = \{T_i\}_{i \in \mathcal{I}}$ of induced subtrees of G , we say that $u \in V(G)$ defines the subset $\hat{u} = \{i \in \mathcal{I} : u \in V(T_i)\}$ of \mathcal{I} .

Lemma 4.1. *Let G be a graph and $\mathcal{T} = \{T_i\}_{i=1}^n$ be a family of $(n-d)$ -vertex induced subtrees of G , any two of which intersect. Suppose that (i) each vertex of G is in exactly $d+1$ members of \mathcal{T} and (ii) for any two vertices $u \neq v$ of G , u and v are together in exactly d members of \mathcal{T} if and only if uv is an edge of G . Then the pure simplicial complex M whose facets are \hat{u} , where $u \in V(G)$, is a neighborly member of $\overline{\mathcal{K}}(d)$, with $\Lambda(M) \cong G$.*

Proof. Let $S \subseteq \{1, \dots, n\}$ be of size d . We show that at most two facets of M contain S . If possible, let \hat{u} , \hat{v} and \hat{w} be three facets of M that contain S . Then by assumption, uv , uw and vw are edges in G . Let $i \in S$. Then by definition of M , u, v, w are vertices of T_i . Since T_i is induced subgraph, we conclude that uv, uw, vw are edges of T_i , which is a contradiction to the fact that T_i is a tree. Thus M is a d -dimensional weak pseudomanifold. Clearly $u \mapsto \hat{u}$ is an isomorphism between G and $\Lambda(M)$. Further the conditions on (G, \mathcal{T}) imply that G should be connected. Thus M is a d -pseudomanifold. Since any two members of \mathcal{T} intersect, it follows that M is neighborly. Let $S_i = \text{st}_M(i)$ be the star of the vertex i in M . Then by construction $\Lambda(S_i) \cong T_i$ and thus $f_d(S_i) = \#(V(T_i)) = n-d$. Also from the neighborliness of M , $f_0(S_i) = n$. Thus $f_0(S_i) = f_d(S_i) + d$ and hence, by Lemma 2.1, S_i is a stacked d -ball. Therefore, by Corollary 2.2, $\text{Lk}_M(i)$ is a stacked $(d-1)$ -ball and hence M is a member of $\overline{\mathcal{K}}(d)$. \square

We use Lemma 4.1 to construct all the complexes. Here we present the details of the construction of A_{41}^5 and $M_{41}^4 = \partial A_{41}^5$.

Construction of A_{41}^5 : Let G be the union of two 41-cycles $C_1 = u_0 u_1 \dots u_{40} u_0$, $C_2 = v_0 v_7 v_{14} \dots v_{34} v_0$ and the paths $P_i = u_i x_i y_i z_i w_i v_i$ for $i \in \mathbb{Z}_{41}$. Consider the family of induced subtrees of G defined by $\mathcal{T} = \{T_i\}_{i=0}^{40}$, where T_i is the subtree induced on G by the following 36 vertices (see Fig 1):

$$\begin{aligned} &u_i, u_{i+1}, \dots, u_{i+5}, v_i, v_{i+7}, \dots, v_{i+35}, x_i, y_i, z_i, w_i, x_{i+2}, y_{i+2}, z_{i+2}, w_{i+2}, x_{i+3}, y_{i+3}, \\ &z_{i+3}, x_{i+4}, y_{i+4}, x_{i+5}, w_{i+14}, w_{i+21}, z_{i+21}, w_{i+28}, z_{i+28}, y_{i+28}, w_{i+35}, z_{i+35}, y_{i+35}, x_{i+35}. \end{aligned}$$

We show that (G, \mathcal{T}) satisfy the conditions in Lemma 4.1 for $d = 5$. From Figure 1, it is easily observed that for $i \in \mathbb{Z}_{41}$,

$$\begin{aligned} \hat{u}_i &= \{i, i-1, i-2, i-3, i-4, i-5\}, & \hat{x}_i &= \{i, i-2, i-3, i-4, i-5, i-35\}, \\ \hat{y}_i &= \{i, i-2, i-3, i-4, i-28, i-35\}, & \hat{z}_i &= \{i, i-2, i-3, i-21, i-28, i-35\}, \\ \hat{w}_i &= \{i, i-2, i-14, i-21, i-28, i-35\}, & \hat{v}_i &= \{i, i-7, i-14, i-21, i-28, i-35\}. \end{aligned}$$

Clearly each vertex of G defines a 6-subset. Further it can be seen that $\hat{x} \cap \hat{y}$ is a 5-element set only for edge pairs like $(\hat{u}_i, \hat{u}_{i+1})$, $(\hat{v}_i, \hat{v}_{i+7})$, (\hat{u}_i, \hat{x}_i) , (\hat{x}_i, \hat{y}_i) etc. Now we show that \mathcal{T} is an intersecting family. First we notice that

$$\varphi := (u_0 \dots u_{40})(x_0 \dots x_{40})(y_0 \dots y_{40})(z_0 \dots z_{40})(w_0 \dots w_{40})(v_0 \dots v_{40})$$

is an automorphism of G and further $\varphi(T_i) = T_{i+1}$ for $i \in \mathbb{Z}_{41}$. Thus we have $T_i = \varphi^i(T_0)$, and so to prove \mathcal{T} to be an intersecting family, it is sufficient to prove that T_0 has non-empty intersection with T_1, \dots, T_{20} . Clearly T_1, \dots, T_5 intersect T_0 in u_1, \dots, u_5

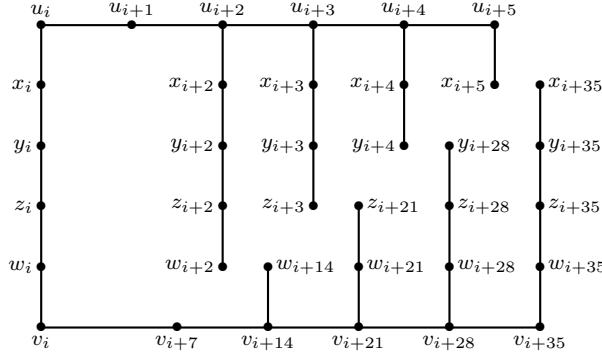


Figure 1: Tree T_i in $G \cong \Lambda(A_{41}^5)$

respectively; T_7, T_{14} intersect T_0 in v_7, v_{14} respectively. Since $6 + 35 = 13 + 28 = 20 + 21 = 0 \pmod{41}$, we see that T_6, T_{13}, T_{20} intersect T_0 in v_0 . Since $8 + 35 = 2 \pmod{41}$ we see that T_8 contains x_2 , which also appears in T_0 . Similarly, $x_3 \in T_9 \cap T_0$, $x_4 \in T_{10} \cap T_0$, $x_5 \in T_{11} \cap T_0$, $w_{14} \in T_{12} \cap T_0$, $y_2 \in T_{15} \cap T_0$, $y_3 \in T_{16} \cap T_0$, $y_4 \in T_{17} \cap T_0$, $z_{21} \in T_{18} \cap T_0$ and $z_{21} \in T_{19} \cap T_0$. Thus, via construction in Lemma 4.1, (G, \mathcal{T}) yields a neighborly member of $\overline{\mathcal{K}}(5)$, which we denote by A_{41}^5 . Finally we note that $\pi: i \mapsto i+1$ is an automorphism of A_{41}^5 by noticing that $\pi(\hat{u}_i) = \hat{u}_{i+1}$, $\pi(\hat{x}_i) = \hat{x}_{i+1}$ etc. This generates the automorphism group \mathbb{Z}_{41} of A_{41}^5 , which indeed is the full automorphism group of A_{41}^5 (checked by `simpcomp`).

Lemma 4.2. *Let $A_{21}^5, B_{21}^5, B_{26}^5, A_{41}^5, M_{21}^4, N_{21}^4, N_{26}^4$ and M_{41}^4 be as in Section 3. Then*

- (a) $A_{21}^5, B_{21}^5, B_{26}^5, A_{41}^5 \in \overline{\mathcal{K}}(5)$,
- (b) $\text{Aut}(A_{21}^5) = \text{Aut}(M_{21}^4) = \text{Aut}(B_{21}^5) = \text{Aut}(N_{21}^4) = \mathbb{Z}_7$,
- (c) $\text{Aut}(B_{26}^5) = \text{Aut}(N_{26}^4) = \mathbb{Z}_{13}$,
- (d) $\text{Aut}(A_{41}^5) = \text{Aut}(M_{41}^4) = \mathbb{Z}_{41}$.

Proof. The properties of the complexes follow from the constructions. As a prototype, we described the construction of A_{41}^5 . The properties of other complexes, mentioned in the statement of the lemma and in Table 1 may be verified by using a combinatorial topology package such as `simpcomp` [5]. For sake of brevity, we omit all the details here. \square

Lemma 4.3. *Let $M_{21}^4, N_{21}^4, N_{26}^4$ and M_{41}^4 be as in Section 3. Then*

- (a) M_{21}^4 and M_{41}^4 are \mathbb{Q} -tight.
- (b) N_{21}^4 and N_{26}^4 are \mathbb{Z}_2 -tight.
- (c) $M_{21}^4, N_{21}^4, N_{26}^4$ and M_{41}^4 are strongly minimal.

Proof. As previously seen M_{21}^4 and M_{41}^4 are triangulations of $(S^3 \times S^1)^{\#8}$ and $(S^3 \times S^1)^{\#42}$ respectively and are in $\mathcal{K}^*(4)$. By Proposition 2.5, they are \mathbb{Q} -tight. Similarly N_{21}^4, N_{26}^4 are triangulations of $(S^3 \times S^1)^{\#8}$ and $(S^3 \times S^1)^{\#14}$ respectively and are in $\mathcal{K}^*(4)$. By Proposition 2.5, they are \mathbb{Z}_2 -tight. By Proposition 2.6, all the complexes here are strongly minimal. \square

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